Philosophy of Set Theory and Foundations

Abstracts

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Toy multiverses of set theory

1 August
10:00
Invited

Victoria Gitman
City University of New York

The philosophical Multiverse View in set theory asserts that the multitude of set theoretical universes studied by set theorists, among them canonical and non-canonical inner models, forcing extensions, ultrapowers, universes with large cardinals, all exist and make up the multiverse of set theory, where no one universe captures the ultimate concept of set. The notion of set in the Multiverse view is relative to each particular universe, so that for example in some universes continuum hypothesis holds, in others the continuum is large, and in others it is weakly inaccessible. The Hamkins Multiverse Axioms attempt to capture a radical position on the relativity of the concept of set. The axioms assert for instance that every universe in the multiverse is a countable set in a larger more powerful universe which moreover thinks that its natural numbers are ill-founded. Together with Hamkins, we showed that if ZFC is consistent, then the collection of all countable computably saturated models of ZFC satisfies all Hamkins' multiverse axioms. Countable computably saturated models - those realizing all their computable types - form a unique natural class with a number of desirable properties such as existence of truth predicates and automorphisms. Indeed, it is not difficult to see that any toy multiverse of countable models satisfying Hamkins' axioms has to consist of some subclass of computably saturated models because any model that is a set in another model of ZFC with ill-founded natural numbers must be computably saturated.

In this talk, I will give an overview of Hamkins' multiverse axioms and the argument that the toy universe of computably saturated models satisfies them. I will also describe a recent modification of Hamkins' axioms studied by Toby Meadows, Michał Godziszewski, Kameryn Williams and myself where we weaken the axiom that every universe must be a set in another universe to say that every universe must be covered by a set in larger universe. I will describe a toy multiverse, satisfying this weaker axiom together with most of the other Hamkins axioms, none of whose models are computably saturated.

Axiomatic theories of truth over set theory, robust realism, and the multiverse

1 Aug
11:15
Contributed

Michał Godziszewski
University of Warsaw

The study of axiomatic truth theories over set theoretical base theories was pioneered by S. Krajewski in [5] who proved the conservativity of $CT^{-}[ZF]$ over ZF.Many years later, his conservativity result was independently refined by A. Enayat and A. Visser in [1], as well as by K. Fujimoto in [2] so as to yield the conservativity of the much stronger theory $CT^{-}[ZF]+Sep^{+}$ over ZF, where Sep^{+} is the natural extension of the separation scheme to formulae with the truth predicate.

In our talk, we will focus on the semantic (model-theoretic) properties of theories of the truth predicate taken with set theory ZF or ZFC taken as the base theory.

The model-theoretic study of truth theories was initiated in the classical papers of S. Krajewski [5] (over arbitrary base theories that include PA and ZF) and H. Kotlarski, S. Krajewski and A. Lachaln [4] (over PA as the base theory). Soon thereafter, in a remarkable paper by A. Lachlan [6], it was shown that if a nonstandard model $M \models PA$ is expandable to a model of $CT^{-}[PA]$, then M is recursively saturated. It can be proved that the same result holds for ω-nonstandard models of ZF, so consequences of Lachlan's theorem impliy that not every model of PA (ZF) is expandable to a model of the compositional truth theory $CT^{-}[PA]$ ($CT^{-}[ZF]$). The above imply together that if $M \models ZFC$ is a countable ω-nonstandard model, then the following are equivalent:

1. M admits an expansion to a model $(M, Tr) \models CT^{-}[ZF]$.
2. $\mathcal{M}$ is recursively saturated.

This characterization, taken together with remarkable construction of V. Gitman and J.D. Hamkins [3] shows that for the class of countable ω-nonstandard models of set theory admitting a compositional truth predicate is equivalent to belonging to the so-called natural model of the Multiverse Axioms.

During the talk we intend to demonstrate the abovementioned results in more detail and explore their philosophical dimensions. In particular, we will demonstrate some details of joint work with T. Meadows, V. Gitman and K. Williams, concerning the constructions of structures satisfying the Weak and the Covering versions of Gitman-Hamkins Multiverse Axioms. Further,we will explore the relations between the research on formal truth theories with philosophy of set theory, focusing on P. Maddy's naturalism w.r.t foundations of mathematics, as described in [7]. Maddy claims that there is no place for arguments employing metaphysical concepts (such as e.g. the concept of truth) in the discussion concerning the reasons for methodological decisions made in the foundations of mathematics. Her argument is that these concepts are inherently robust-realistic, which makes them unsuitable for naturalistic thinking about foundations, since the latter does not allow for reasons involving strong metaphysical commitments. I will try to argue that contrary to the model-theoretic ways of characterizing the concept of mathematical truth (such as Tarski's definition or Kripke's construction), if we use the axiomatic approach to characterizing mathematical truth (i.e. where the notion of truth treated as a primitive undefined predicate rather than defined in model-theoretic terms), then the use of the concept of truth does not necessarily lead to robust-realistic commitments in philosophy of mathematics. In particular, we will argue that treating the notion of compositional truth (as axiomatized in CT^{-}) as the notion of mathematical truth simpliciter obeys the principles of Maddy's naturalism in foundations of mathematics, and allows for an essentially truth-theoretic argument in favour of pluralism in philosophy of set theory.

Last, but not least, we will also show some relevant properties of models of the so-called disquotational theories of truth (such as the so-called locally disquotational theory TB) over set theories, which has some philosophical implications in the debate on deflationism w.r.t. the concept of mathematical truth.

References

[1] Ali Enayat and Albert Visser. *Full satisfaction classess in a general setting*. Unpublished manuscript, 2018.

[2] Kentaro Fujimoto. *Classes and truths in set theory*. **Annals of Pure and Applied Logic**, 163:1484-1523, 2012.

[3] Victoria Gitman and Joel David Hamkins. *A natural model of the multiverse axioms*. **Notre Dame Journal of Formal Logic**, 51, 10 2010.

[4] Henryk Kotlarski, Stanislaw Krajewski, and Alistair Lachlan. *Construction of satisfaction classes for nonstandard models*. **Canadian Mathematical Bulletin**, 24(3):283-293, 1981.

[5] Stanislaw Krajewski. *Non-standard satisfaction classes*. In W. Marek et al., editor - **Set Theory and Hierarchy Theory. A Memorial Tribute to Andrzej Mostowski**, pages 121-144. Springer, 1976.

[6] Alistair H. Lachlan. *Full satisfaction classes and recursive saturation*. **Canadian Mathematical Bulletin**, 24:295-297, 1981.

[7] Penelope Maddy. *Set-theoretic foundations*, in: Andres Eduardo Caicedo, James Cummings, Peter Koellner, Paul B. Larson - **Foundations of Mathematics: Logic at Harvard Essays in Honor of W. Hugh Woodin? 60th Birthday March 27-29 2015, Harvard University, Cambridge, MA**, pages 289-322, AMS 2017.

Mathematical foundations and (meta-)mathematical language

1 Aug
13:00
Contributed

Henning Heller
Universität Wien

In Landry (1999), the author claims that “category theory provides a *language* for mathematics” in that category theory provides a framework to talk about mathematical structures without determining a meaning or content to them. In my presentation, I want to clarify whether Landry’s claim can serve as an argument for category-theoretic foundations.
The discussion follows Maddy (2017) who argues that any argument for or against some specific foundational program must first clarify its exact intended *foundational use*. Hence the first question is whether the claim “X provides a language for mathematics” provides a foundational use of theory X. At first glance, a positive answer to this question seems reasonable. But Landry herself provides arguments against foundationalism in general and category-theoretic foundations in particular, and a closer examination is necessary.
Once the foundational use “X provides a language for mathematics” is settled, I investigate under which conditions category theory actually fits this claim. Landry argues that the deliberate indeterminacy of the category-theoretic language enables it to provide a formal framework not only about mathematical structures, but also about the very notion of structure itself. In other words, category theory does not only found (parts of) mathematics, but also (parts of) meta-mathematics, it exhibits a (foundational) double role. A similar argument is employed in Corry (2004) who emphasizes the *reflexive* character of category theory as a theory to account for both *body* and *image* of modern mathematics. Although both Landry’s and Corry’s arguments stem from a structuralist perspective and are therefore vulnerable to a number of additional objections, it is not clear “how much” of structuralism is really necessary to uphold

their arguments.

Finally, it will be interesting to analyze whether set theory (both the universe and the multiverse conception) may also fulfill the foundational use of "providing a language" for mathematics. Set theory surely *interprets* all of mathematics, but might provide the same indeterminacy towards content and meaning of mathematics. Therefore set theory might turn out too "rigid" to account for the "protean" (Mac Lane, 1986) character of mathematics.

[1] Leo Corry. *Modern Algebra and the Rise of Mathematical Structures*. Birkhäuser, 2004.

[2] Elaine Landry. "Category theory: The language of mathematics". In: *Philosophy of Science* 66 (1999), S14–S27.

[3] Saunders Mac Lane. *Mathematics Form and Function*. Springer New York, 1986.

[4] Penelope Maddy. "Set-theoretic foundations". In: *Foundations of Mathematics*. Ed. by Andrés Eduardo Caicedo. Vol. 690. American Mathematical Soc., May 2017, p. 289.

On the extent of intrinsic justifications for large cardinal axioms in set theory

1 Aug
13:50
Contributed

Rupert McCallum

Universität Tübingen

In earlier work we have formulated new large-cardinal axioms of strength intermediate between totally indescribable cardinals and ω-Erdős cardinals, which turn out to be connected with the virtual large cardinals that have been studied by Ralf Schindler and Victoria Gitman and others. We made arguments that these axiom should be seen as intrinsically justified. Welch and Roberts have recently put forward a family of reflection principles, Welch's principle implying the existence of a proper class of Shelah cardinals and provably consistent relative to a superstrong cardinal, and Roberts' principle implying the existence of a proper class of 1-extendible cardinals and provably consistent relative to a 2-extendible cardinal. Roberts tentatively argued that his principle should be seen as intrinsically justified (at least on the assumption that a weaker form of reflection involving reflection of second-order formulas with a second-order parameter should be seen as intrinsically justified). This work overlapped with previous work of Victoria Marshall's on reflection principles. We will discuss the relationship between reflection principles equivalent to those studied in our own earlier work and stronger but similar reflection principles which are natural extensions of those of Welch and Roberts, and show how the notion of a virtual large cardinal arises naturally in this context. We will also show how a natural strengthening of Roberts' reflection principle yields the existence of supercompact cardinals, and examine how ideas presented in an earlier paper of Victoria Marshall might be used to motivate still stronger reflection principles yielding still stronger large cardinals. We shall explain how this led us to formulate a new large-cardinal axiom positing the existence of what we call a "hyper-enormous" cardinal, which may be consistent with ZFC and is of greater consistency strength than anything previously considered short of the choiceless cardinals, and describe some of the applications of this concept.
